JOURNAL OF APPROXIMATION THEORY **28**, 30–35 (1980)

Note on Mean Convergence of Lagrange Parabolas

P. VÉRTESI

Mathematical Institute of the Hungarian Academy of Sciences, 1053 Budapest, Reáltanoda u. 13–15, Hungary

Communicated by Richard S. Varga

Received July 6, 1978

1. INTRODUCTION AND PRELIMINARY RESULTS

1.1. Let $w(x)$ be a weight function on $[-1, 1]$ and let $p_n(w; x)$ be the sequence of the corresponding orthogonal polynomials with the zeros

$$-1 < x_{nn}(w) < x_{n-1,n}(w) < \cdots < x_{1n}(w) < 1 \qquad (n = 1, 2,...).$$

For $f \in C$ (f is continuous on $[-1, 1]$) we consider the uniquely determined Lagrange interpolatory polynomials $L_n(f; w; x) = \sum_{k=1}^{n} f(x_{kn}(w))\, l_{kn}(w; x)$ of degree $\leqslant n - 1$ satisfying $L_n(f; w; x_{kn}) = f(x_{kn})(k = 1, 2,..., n; n = 1, 2,...)$. As was proved by G. Faber, $L_n(f; w; x)$ does not necessarily converge uniformly to $f(x)$. But as far as mean convergence is concerned, the situation is more favorable.

A general theorem due to Erdös and Turán [1] states that

$$\lim_{n\to\infty} \int_{-1}^{1} [L_n(f; w; x) - f(x)]^2\, w(x)\, dx = 0 \qquad \text{if} \quad f \in C. \tag{1.1}$$

As to Jacobi weights, A. Holló and later Turán [2] proved that

$$\lim_{n\to\infty} \int_{-1}^{1} [L_n(f; \alpha, \beta; x) - f(x)]^2\, dx = 0 \qquad \textit{if} \quad f \in C, \tag{1.2}$$

provided $-1 < \alpha, \beta < \frac{1}{2}$; *moreover*

$$\lim_{n\to\infty} \int_{-1}^{1} | L_n(f; \alpha, \beta; x) - f(x)|\, dx = 0 \qquad \textit{if} \quad f \in C, \tag{1.3}$$

provided $-1 < \alpha, \beta < \frac{3}{2}$.

(Here and later, $L_n(f; \alpha, \beta; x)$ stands for $L_n(f; w; x)$ where $w(x) = w^{(\alpha,\beta)}(x) = (1 - x)^\alpha(1 + x)^\beta$ $(\alpha, \beta > -1)$ is a Jacobi weight.)

0021-9045/80/010030–06$02.00/0

1.2. A far-reaching generalization of (1.2) and (1.3) was proved by Askey [3, 4] and later by Névai [5]. The following is essentially a result from [5]:

If

(i) $-1 < \alpha, \beta \leqslant -\frac{1}{2}$; $a = b = 0$ *and* $p > 0$ *or*

(ii) $-\frac{1}{2} < \alpha, \beta$; $a > (2\alpha - 3)/4$, $b > (2\beta - 3)/4$ *and*

$$0 < p < \min\left(\frac{4(a+1)}{2\alpha+1}, \frac{4(b+1)}{2\beta+1}\right) \qquad or$$

(iii) $-1 < \alpha \leqslant -\frac{1}{2} < \beta$; $a = 0$, $b > (2\beta - 3)/4$, *and*

$$0 < p < \frac{4(b+1)}{2\beta+1} \qquad or$$

(iv) $-1 < \beta \leqslant -\frac{1}{2} < \alpha$; $b = 0$, $a > (2\alpha - 3)/4$ *and*

$$0 < p < \frac{4(a+1)}{2\alpha+1};$$

then

$$\lim_{n\to\infty} \int_{-1}^{1} | L_n(f; \alpha, \beta; x) - f(x)|^p (1-x)^a (1+x)^b \, dx \qquad 0 \qquad \text{for any} \quad f \in C.$$

Conversely, if $\alpha > -\frac{1}{2}$ *and* (1.4) *holds, then necessarily* $p \leqslant 4(a+1)/(2\alpha+1)$. *Similarly, if* $\beta > -\frac{1}{2}$ *and* (1.4) *is true, we have* $p \leqslant 4(b+1)/(2\beta+1)$.

1.3. Of course, (1.2) (or (1.3)) can be obtained from (1.4) if we choose $a = b = 0, p = 2$ (or $a = b = 0, p = 1$).

2. A New Result

2.1. A natural problem which was raised in [2] and [5] is to study the limiting cases, i.e., when $a = b = 0, p = 2$, and $\max(\alpha, \beta) = \frac{1}{2}$ (see (1.2)); when $a = b = 0$, $p = 1$, and $\max(\alpha, \beta) = \frac{3}{2}$; when (ii) is modified to read $p = \min\{4(a+1)/(2\alpha+1), 4(b+1)/(2\beta+1)\}$; or when, in (iii), $p = 4(b+1)/(2\beta+1)$ (or $p = 4(a+1)/(2\alpha+1)$ in (iv)). The only settled case is $a = b = 0$, $p = 2$, $\alpha = \beta = \frac{1}{2}$ (see Feldheim [6]), in which case (1.2) does not hold for all $f \in C$.

2.2. Let $C(\omega) = \{f; f \in C$ and $\omega(f; t) \leqslant a(f)\, \omega(t)\}$ where $\omega(f; t)$ is the modulus of continuity of $f(x)$ and $\omega(t)$ is a modulus of continuity with $\lim_{t=0} t/\omega(t) = 0$.

THEOREM 2.1. *Let* $a, b, \alpha, \beta > -1$, $p \geqslant 1$, *and* $0 \leqslant \epsilon < 2$ *be fixed numbers. If*

$$-\frac{1}{2} < \alpha \qquad \text{and} \qquad p = \frac{4(a+1)}{2\alpha+1} + q \quad (q \geqslant 0), \tag{2.1}$$

then for certain $f \in C(\omega)$ *and* $n = n_1, n_2, \ldots,$

$$\begin{aligned} \int_{-1+\epsilon}^{1} | L_n(f; \alpha, \beta; x) - f(x)|^p (1-x)^a (1+x)^b \, dx & \\ > \left[\omega\left(\frac{1}{n^2}\right)\right]^p \log n \qquad & \text{if} \quad q = 0, \\ > \left[\omega\left(\frac{1}{n^2}\right)\right]^p n^{q(\alpha+1/2)} \qquad & \text{if} \quad q > 0. \end{aligned} \tag{2.2}$$

Similarly, if

$$-\frac{1}{2} < \beta \qquad \text{and} \qquad p = \frac{4(b+1)}{2\beta+1} + q \quad (q \geqslant 0), \tag{2.3}$$

then for certain $f \in C(\omega)$ *and* $n = n_1, n_2, \ldots,$

$$\begin{aligned} \int_{-1}^{1-\epsilon} | L_n(f; \alpha, \beta; x) - f(x)|^p (1-x)^a (1+x)^b \, dx & \\ > \left[\omega\left(\frac{1}{n^2}\right)\right]^p \log n \qquad & \text{if} \quad q = 0, \\ > \left[\omega\left(\frac{1}{n^2}\right)\right]^p n^{q(\beta+1/2)} \qquad & \text{if} \quad q > 0. \end{aligned} \tag{2.4}$$

2.3. Theorem 2.1 implies that, in the above limiting cases, the relations corresponding to (1.2)–(1.4) do not hold for all $f \in C$.

3. PROOF

3.1. Let us define $f_n(x) \in C$ as follows:

$$\begin{aligned} f_n(x_{kn}^{(\alpha,\beta)}) &= (-1)^k \qquad (k = 0, 1, 2, \ldots, n+1), \\ f_n(x) \text{ is linear in } &[x_{kn}^{(\alpha,\beta)}, x_{k-1,n}^{(\alpha,\beta)}] \qquad (k = 1, 2, \ldots, n+1). \end{aligned} \tag{3.1}$$

Here, sometimes omitting superfluous indices, $x_0 = 1$, $x_{n+1} = -1$; $x_{kn} = \cos\theta_{kn}$ $(k = 1, 2, \ldots, n)$ stand for the zeros of the Jacobi polynomial

$P_n(x) \equiv P_n^{(\alpha,\beta)}(x)$ of degree n with the normalization $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{\alpha}$. Assuming

$$-\frac{1}{2} < \alpha \quad \text{and} \quad p = \frac{4(a+1)}{2\alpha+1} + q \quad (q \geqslant 0), \tag{3.2}$$

our fundamental lemma is

LEMMA 3.1.

$$\begin{aligned} \int_{-1+\epsilon}^{1} | L_n(f_n ; x)|^p (1-x)^a (1+x)^b \, dx &> c_1 \log n \qquad \text{if} \quad q = 0, \\ &> c_2 n^{q(\alpha+1/2)} \qquad \text{if} \quad q > 0. \end{aligned} \tag{3.3}$$

3.1.1. To prove (3.3) let us denote by x_j a zero of P_n nearest to x $(1 \leqslant j = j(n, x) \leqslant n)$. Setting

$$\lambda_n(x) = \sum_{k=1}^{n} | l_{kn}(x)|,$$

$$s_n = [n(\log n)^{-2/(\alpha+1/2)}], \qquad \text{and} \qquad x = \cos\theta,$$

we have

$$| L_n(f_n ; x)| \sim \lambda_n(x) \sim \theta_j^{-\alpha-1/2} \sim \theta^{-\alpha-1/2} \qquad \text{if} \quad \theta \in T_j \text{ and } 3 \leqslant j \leqslant s_n , \tag{3.4}$$

where

$$T_j = \left[\frac{\theta_{j+1} + \theta_j}{2}, \frac{\theta_{j+1} + 3\theta_j}{4}\right].$$

(As to the symbol "$\sim$" which does not depend on x, see [7, 1.1].)

Indeed, by (3.1),

$$\begin{aligned} | L_n(f_n ; x)| &= \left| \sum_{k=1}^{n} (-1)^k l_k(x) \right| \\ &= \left| \sum_{k=1}^{j-1} + \sum_{k=j+1}^{n} + \sum_{k=j} \right| \overset{\text{def}}{=} | J_1 + J_2 + J_3 |. \end{aligned} \tag{3.5}$$

By [8, Lemmas 3–5], we can write

$$| J_1 | \sim | P_n(x)| \, n^{-\alpha} I_1 \qquad \text{and} \qquad | J_2 | \sim | P_n(x)| \, n^{-\alpha} I_2 , \tag{3.6}$$

where

$$I_1 \sim j^{\alpha+1/2} \log j \qquad \text{and} \qquad I_2 \sim j^{\alpha+1/2} \log j + n^{\alpha+1/2}. \tag{3.7}$$

(We used the fact that the summands of J_s, $s = 1, 2$, have the same sign.) As $j \leqslant s_n$, we can write $I_2 \sim n^{\alpha+1/2}$ and $I_1 = o(I_2)$. By [9, Lemma 3.2],

$$| P_n(x)| \sim | \theta - \theta_j | \theta_j^{-\alpha-1/2} n^{1/2} \sim | x - x_j | \theta_j^{-\alpha-3/2} n^{1/2}; \qquad (3.8)$$

hence, $| J_3 | \sim 1$, using [7, (8.9.2)] which states that

$$| P_n'(x_k)| \sim k^{-\alpha-3/2} n^{\alpha+2} \qquad (k = 1, 2,..., n). \qquad (3.9)$$

From (3.5)–(3.8) we get $| L_n(f_n ; x)| \sim \theta_j^{-\alpha-1/2}$ by applying the relation

$$\theta_{k+1} - \theta_k \sim \frac{1}{n} \qquad (k = 0, 1, 2,..., n). \qquad (3.10)$$

(Here $\theta_0 = 0$, $\theta_{n+1} = \pi$; see, e.g., [9, Lemma 3.1].) Using [8, Lemma 5], we obtain (3.4), observing that, for $\theta \in T_j$, one has $\theta \sim \theta_j$.

3.1.2. Using the substitutions $x = \cos \theta$, $-1 + \epsilon = \cos \delta$ and (3.4) we can write:

$$\begin{aligned}
&\int_{-1+\epsilon}^{1} | L_n(f_n ; x)|^p (1 - x)^a (1 + x)^b \, dx \\
&\quad \geqslant c \int_{-1+\epsilon}^{1} | L_n(f_n ; x)|^p (1 - x)^a \, dx \\
&\quad = c \int_0^{\delta} | L_n(f_n ; \cos \theta)|^p \left(\sin \frac{\theta}{2}\right)^{2a} \sin \theta \, d\theta \\
&\quad \geqslant c \sum_{j=3}^{s_n} \int_{T_j} | L_n(f_n ; \cos \theta)|^p \left(\sin \frac{\theta}{2}\right)^{2a} \sin \theta \, d\theta \\
&\quad \sim \sum_{j=3}^{s_n} \int_{T_j} \frac{\theta^{2a+1}}{\theta^{(\alpha+1/2)[4(a+1)/(2\alpha+1)+q]}} \, d\theta = \sum_{j=3}^{s_n} \int_{T_j} \frac{d\theta}{\theta^{1+q(\alpha+1/2)}} \\
&\quad \sim n^{q(\alpha+1/2)} \sum_{j=3}^{s_n} \frac{1}{j^{1+q(\alpha+1/2)}} \sim \log n \qquad \text{if} \quad q = 0, \\
&\qquad\qquad\qquad\qquad \sim n^{q(\alpha+1/2)} \qquad \text{if} \quad q > 0,
\end{aligned}$$

which is (3.3).

3.2. Now we can apply the method of [10].

Indeed, choosing in [10, 2.4] $m = e_n = 1$, $g_n = f_n$, $T_n(g_n ; z_n) = (\int_{-1+\epsilon}^{1} | L_n(f_n ; x)|^p (1 - x)^a (1 + x)^b)^{1/p}$, $\lambda_n(z_n) = (\log n)^{1/p}$ (or $\lambda_n(z_n) = n^{q(\alpha+1/2)/p}$, if $q > 0$), $\delta_n = n^{-2}$, $U_n(g_n; z_n) = (\int_{-1+\epsilon}^{1} | g_n(x)|^p (1 - x)^a (1 + x)^b)^{1/p} n$

and $T_n(h; x) - U_n(h; x) = (\int_{-1+\epsilon}^{1} | L_n(h; x) - h(x)|^p(1 - x)^a(1 + x)^b\,dx)^{1/p}$, we obtain, as in [10], that for certain $f \in C(\omega)$ and $\{n_i\}$,

$$\left(\int_{-1+\epsilon}^{1} | L_n(f; x) - f(x)|^p(1 - x)^a(1 + x)^b\,dx\right)^{1/p} > \omega\left(\frac{1}{n^2}\right)(\log n)^{1/p} \tag{3.11}$$

if $q = 0$ and $n = n_1, n_2, \ldots$, from which (2.2) follows. The remaining cases can be treated similarly.

References

1. P. Erdös and P. Turán, On interpolation, I, *Ann. of Math.* **38** (1937), 142–155.
2. P. Turán, On some problems in the theory of the mechanical quadrature, *Mathematica (Cluj)* **8** (1966), 182–192.
3. R. Askey, Mean convergence of orthogonal series and Lagrange interpolation, *Acta Math. Acad. Sci. Hungar.* **23** (1972), 71–85.
4. R. Askey, Summability of Jacobi series, *Trans. Amer. Math. Soc.* **179** (1973), 71–84.
5. G. P. Névai, Mean convergence of Lagrange interpolation, I, *J. Approximation Theory* **18** (1976), 363–377.
6. E. Feldheim, Quelques recherches sur l'interpolation de Lagrange et d'Hermite par la méthode du développement des fonctions fondamentales, *Math. Z.* **44** (1938), 55–84.
7. G. Szegö, "Orthogonal Polynomials," Vol. XXIII, Amer. Math. Soc. Coll. Publ., New York, 1959.
8. G. I. Natanson, Two-sided estimate for the Lebesgue function of the Lagrange interpolation with Jacobi nodes, *Izv. Vysš. Učebn. Zaved. Matematika* **11** (1967), 67–74 [Russian].
9. P. Vértesi, On Lagrange interpolation, *Period. Math. Hungar.*, in press.
10. P. Vértesi, On certain linear operators, VII, *Acta Math. Acad. Sci. Hungar.* **25** (1974), 67–80.